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# Non-local barrier and anisotropy of alpha emissions from oriented $^{237}$Np nuclei

M. L. CHAUDHURY

Department of Physics and Mathematics, Indian School of Mines, Dhanbad, Bihar, India

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**Abstract.** A formula for the angular distributions of alpha particles from oriented nuclei is presented here by assuming the barrier to be the usual *anisotropic* electrostatic potential superimposed by a non-local alpha–nuclear interaction potential. A spherically symmetric form of this latter potential was used successfully in a previously published work and will now be taken to be *axially* symmetric, described by a deformation parameter $\epsilon_l$. For simplicity we shall restrict our considerations to spheroidal deformations of the nuclear surface. Calculations show that the anisotropy coefficient is *positive* for either sign of a given value of the quadrupole moment $Q$ of the parent nucleus, depending on the values of $\epsilon_l$. The significance of this finding in the light of the results of experiments on $^{237}$Np nuclei is discussed. The main contribution of this work is to show that the observed anisotropy of alpha emissions from oriented $^{237}$Np nuclei can now be interpreted without difficulty, regardless of the sign of $Q$ of this nuclide.

## 1. Introduction

It was suggested by Hill and Wheeler (1953) that, for prolate spheroidal nuclei, the electrostatic barrier being thinner and lower at the poles than at the nuclear equator, preferential $\alpha$-emissions should occur along the nuclear symmetry axis. Meanwhile, studies in $\alpha$-ray spectroscopy (cf. Perlman and Asaro 1954) for very heavy elements revealed rotational level structures in the daughter nuclei in accordance with the Rainwater (1950) and Bohr–Mottelson model (Bohr 1952, Bohr and Mottelson 1953) of strongly coupled nucleonic structure of deformed nuclei. Hereafter the development of $\alpha$-decay theories proceeded along these lines in rapidly succeeding papers, for example, by Christy (1955), Bohr *et al.* (1955), Rasmussen and Segall (1956), Rose (1957), Fröman (1957), Strutinskii (1957) and Rostovskii (1961), by taking into account various effects of deformations of the nuclear surface as well as of angular momentum. In this early period the discussions on $\alpha$-decay intensities were mainly restricted to the hypothesis of a pure Coulomb potential barrier. Although alpha angular distributions are not always explicitly calculated, it is found that if the barrier is assumed to be purely electrostatic then the conclusions are in general agreement with the prediction of Hill and Wheeler.

On the other hand, Mang (1960) and Zeh and Mang (1962) in recent papers have treated the $\alpha$-decay problem in terms of clustering probability by using an angle-dependent shell-model wave function on the nuclear surface. As for the potential barrier, the usual Coulomb potential plus an $\alpha$–nuclear interaction potential has been taken. Also, as discussed below, there are reasons to believe that the said $\alpha$–nuclear interaction potential is non-local in character. In Mang's theory this potential has been taken to be *static*. However, it must be stated that the shell-model theory does not indicate anything to support the static potential assumed therein. Furthermore, as also mentioned in their latest paper (Poggenburg *et al.* 1969), the agreement between the shell-model calculations and the empirical hindrance factors is unsatisfactory for higher $\alpha$-angular momenta. Hence the need arises for reviewing the situation on the basis of a more realistic barrier than either the purely electrostatic barrier or a *static* $\alpha$–nuclear interaction potential. This is all the more important for the present problem of $\alpha$-angular distributions in view of the difficulty that arises in interpreting the results of experiments on $^{237}Np$ nuclei on the basis of the usual barrier hypothesis.

The measurement of the anisotropy of $\alpha$-emissions was carried out by Hanauer *et al.* (1961) confirming the results of their previous paper (see, e.g. Dabbs *et al.* 1958). In their experiments $^{237}Np$ nuclei were oriented through large electric quadrupole and magnetic hyperfine couplings in the $NpO_2^{2+}$ group of the salt $NpO_2Rb(NO_3)_3$. It was found that the $\alpha$-particles are emitted preferentially *at right angles* to the crystalline *c* axis. However, the interpretation of the orientation of the symmetry axes of the $^{237}Np$ nuclei relative to this crystal axis depends on the signs of the hyperfine coupling constants $A$ and $P$. Hanauer and co-workers determined their signs to be $A < 0$ and $P > 0$. Pryce (1959) also in an earlier work had come to the same conclusion. This means that the $^{237}Np$ nuclei were oriented *at right angles* to the crystalline *c* axis, thereby indicating the preferential $\alpha$-emissions to be *along* the nuclear angular momentum vector. Eisenstein and Pryce (1955) have shown that, if $P > 0$, then the sign of the electric quadrupole moment $Q$ is negative, i.e. $^{237}Np$ nuclei are *oblate*. Thus, according to this interpretation, the $\alpha$-emissions in the above experiments were actually observed to be more intense at the 'flat' surfaces than at the 'tips' of the nuclei, in contradiction to the expectation from the pure Coulomb barrier hypothesis.

In view of this difficulty, Hanauer and co-workers assumed a positive value for $Q$ in $^{237}Np$ nuclei and, in order to reconcile this with the finding that $P > 0$, they assumed for the electronic structure of the neptunyl ions a model of bonding which is different from that suggested by Bleaney *et al.* (1954). At present there is perhaps no other evidence to settle this point in favour of either opinion, except that a positive value of $Q$ for $^{237}Np$ nuclei is necessary if the barrier is assumed to be purely Coulombic.

On the other hand, the reasons for considering that a purely electrostatic barrier in $\alpha$-decay theory is an oversimplification and should be modified by an appropriate superimposed $\alpha$–nuclear potential, have been discussed earlier (Chaudhury 1952). It was then proposed (Chaudhury 1960) that the said $\alpha$–nuclear interaction potential should be non-local, in view of the observed momentum dependence of the nuclear force as evidenced in different experiments (see, e.g. Igo and Thaler 1957, Weisskopf 1957 and Wilson and Sampson 1965). Question might arise as to whether or not the effects of non-locality on $\alpha$-decay intensities are negligible. On the contrary it is found (Chaudhury 1963) that if the form of non-locality for the $\alpha$–nuclear potential is taken to be the same as that assumed for the nucleon–nuclear interaction potential (cf., for example, Frahn and Lemmer 1957, Lemmer and Green 1960) then it leads to rather too *large* a change in the barrier penetration factor. It thus appears that the momentum dependence associated with the $\alpha$–nuclear potential is somewhat smaller than envisaged in the effective-mass approximation in the nucleon–nuclear potential suggested by Frahn. This was also noted by other authors (see, e.g. Preston 1962). From these considerations a simpler form of non-local $\alpha$–nuclear potential, within the framework of the optical model, was defined (Chaudhury 1966—to be referred to as I) in terms of which relative intensities of $\alpha$-spectra, particularly those for the odd-parity transitions (not previously explained) in spherical nuclei in the range $83 \leqslant Z \leqslant 92$ and neutron number $N \leqslant 138$, were discussed. Apart from the simplicity of the treatment, it is clearly shown that, for the spherical nuclei, the data for the relative intensities can be very well represented in this way, agreements being in most cases within 25% of the observed values.

Corresponding results with a pure Coulomb potential or the said static potential show much wider discrepancies.

In view of this success and recalling the somewhat unclear experimental situation, we intend to discuss here the problem of $\alpha$-angular distributions on the basis of the modified barrier mentioned above. Now the anisotropy of $\alpha$ emissions may be partly determined by an angle-dependent $\alpha$-wave function on the nuclear surface. It is very improbable that the $\alpha$-wave function on the surface is so distorted as to overcome the effects of the barrier. It is therefore expected that barrier effects will be the dominating influence in determining the character of the anisotropy of $\alpha$-emissions from oriented nuclei.

For simplicity we shall restrict our considerations to spheroidal shapes of the nuclear surface determined by the quadrupole deformation parameter $\beta_{2,0}$. To be consistent, the *isotropic* non-local potential defined in the earlier paper I will now be extended and described by a deformation parameter, say $\epsilon_l$. Since the potential itself is momentum dependent, the parameter $\epsilon_l$ may take different values for different $\alpha$-angular momenta $l$. Since in the present problem we shall be mainly concerned with the ground state transitions, the subscript of $\epsilon$ will be taken as $l = 0$. In § 2, the relevant wave equation is set up on the basis of the above assumptions. This is then solved by following a method due to Christy, under the boundary conditions that the $\alpha$-wave function $\psi_0(\theta, \varphi)$ on the nuclear surface is represented as an expansion in spherical harmonics. In the next section, the desired formula for the anisotropy coefficient is obtained.

Now in view of the differences of opinion about the sign of $Q$ for $^{237}$Np nuclei, the formula for the $\alpha$-anisotropy cannot be compared quantitatively with the experimental results. Calculations are therefore made for both the positive and negative signs of a probable value of $Q$ for $^{237}$Np nuclei. Also, since the non-local deformation parameter $\epsilon_0$ is unknown, a plausible range of values of $\epsilon_0$ is chosen for each set of calculations. As we are mainly interested in the *sign* of the anisotropy coefficient, it is sufficient to retain in the formula only the basic term in the series of $\psi_0(\theta, \varphi)$, representing the $\alpha$-wave function on the nuclear surface. It will, however, be shown that our conclusions are unaffected if the quadrupole terms of $\psi_0(\theta, \varphi)$ are also included. In the last section a brief discussion and conclusions are given.

It will be seen from the tabulated results that the experiments of Hanauer *et al.* (1961) can now be interpreted without difficulty and regardless of the sign of $Q$ of the $^{237}$Np nuclei. Secondly, indications about the limiting values of the parameter $\epsilon_0$ are available. This information will be useful for extending the present considerations to the problem of intensities of $\alpha$-spectra for deformed nuclei.

## 2. Wave equation

### 2.1. *Isotropic barrier*

From what has been said above, if one starts from the Schrödinger equation for an *isotropic* barrier

$$\left\{-\frac{h^2}{2\mu}\nabla^2 + u_0(r) + V_{\text{non-local}}\right\}\psi(\boldsymbol{r}) = E\,\psi(\boldsymbol{r}) \tag{1}$$

then, for the partial $\alpha$-wave of angular momentum $l$,

$$\psi(\boldsymbol{r}) = r^{-1}\, U_l(r)\, Y_{l,m}(\theta, \varphi)$$

one obtains the equation

$$\frac{\hbar^2}{2\mu}\left\{\frac{d^2}{dr^2}U_l - \frac{l(l+1)}{r^2}U_l\right\} + \{E_l - u_0(r)\}U_l = \int \chi_l(r, r'')\, U_l(r'')\, dr'' \tag{2}$$

where $u_0 = 2(Z-2)e^2/r$ ($Z$ being the charge number of the parent nucleus), $e$ is electronic charge, $\boldsymbol{r}''(= r'', \theta'', \varphi'')$ represents some position of the $\alpha$-particle other than that denoted by $\boldsymbol{r}(= r, \theta, \varphi)$ in the laboratory system of coordinates with the origin at the centre of mass of the two-body system, and $\chi_l$ is the angle-independent part of the interaction kernel $J(\boldsymbol{r}, \boldsymbol{r}'')$.

For the nucleon–nuclear potential, the form of $J(\boldsymbol{r}, \boldsymbol{r}'')$, was suggested by Frahn and Lemmer (1957) as

$$J(\boldsymbol{r}, \boldsymbol{r}'') = V\left(\left|\frac{\boldsymbol{r}+\boldsymbol{r}''}{2}\right|\right)\delta_b(\boldsymbol{r}-\boldsymbol{r}''). \tag{2.1}$$

As already mentioned the same form is not applicable for the $\alpha$-nuclear potential. Hence $J(\boldsymbol{r}, \boldsymbol{r}'')$ was defined (cf. I) to suit $\alpha$-decay data as

$$J(\boldsymbol{r}, \boldsymbol{r}'') = V(r)\,\delta_b(\boldsymbol{r}-\boldsymbol{r}''). \tag{3}$$

In equation (3) the static part $V(r)$ will be taken as before to be the real part of the optical model potential and $\delta_b$ is represented by a Gaussian function $\pi^{-3/2}b^{-3}\exp\{(\boldsymbol{r}-\boldsymbol{r}'')^2/b^2\}$, $b$ being the range of non-locality. It must be stated that the optical parameters are not unique (cf., for example, Wilson and Sampson 1965). Also, scattering and decay mechanisms are of course not identical. Thus for lack of information one has to choose any one of the sets of parameters indicated from other sources. Since in I the Igo potential was found suitable, the same will be used here.

Now by a Taylor expansion of $U_l(r'')$ around $r$, the integral in equation (2) gives the non-local energy as a series expansion

$$U_l(r)\int \chi_l(r, r'')\, dr'' + U_l'(r)\int (r''-r)\,\chi_l(r, r'')\, dr'' + U_l''(r)\int \frac{(r''-r)^2}{2}\chi_l(r, r'')\, dr'' + \ldots. \tag{3.1}$$

In I for simplicity we retained only the first term in the series (3.1). In this approximation one would probably prefer to call the potential an $l$-dependent one rather than a full non-local potential. However, in the present discussion we remove this approximation because higher-order terms will now arise from the quadrupole moment of the nuclear surface. Integrating equation (3.1) and retaining terms up to the order $b^2$, it can easily be shown that the non-local interaction energy in general takes the form

$$v_l^{(n)}(r) = V(r)\, p_l^{(n)}(r) \tag{4}$$

where the purely non-local part of order $n$ (shown as superscript) is given by

$$p_l^{(n)}(r) = g^{(n)}(z)\left[1 - \frac{l(l+1)}{4r^2}\left\{1 - \frac{g^{(2)}(z)}{g^{(0)}(z)}\right\}\right] \tag{4.1}$$

where $z = (r-R_i)/b$, $R_i$ is the 'inner turning point' and

$$\begin{aligned} g^{(0)}(z) &= \tfrac{1}{2}\left(1+2\pi^{-1/2}\int e^{-z^2}\,dz\right) \\ g^{(1)}(z) &= \tfrac{1}{2}\left(1-\pi^{-1/2}\int_{\sigma} e^{-z^2}\,z\,dz\right) \\ g^{(2)}(z) &= \tfrac{1}{2}\left(1+4\pi^{-1/2}\int e^{-z^2}\,z^2\,dz\right). \end{aligned} \tag{4.2}$$

After simplification, equation (2) can be written from equations (3) and (4.1) as

$$\begin{aligned} &\frac{\hbar^2}{2\mu}\left\{U_l''(r)-\frac{l(l+1)}{r^2}\,U_l(r)\right\}+\{E_l-u_0(r)-v_l^{(0)}(r)\} \\ &\quad\times\left\{1+\frac{b^2\mu}{2\hbar^2}\,v_0^{(2)}(r)\right\}U_l(r) = b\pi^{-1/2}\,v_0^{(1)}(r)\;U_l'(r) \end{aligned} \tag{5}$$

where primes refer to differentiations with respect to $r$. Now the right-hand side of (5) is a small perturbation, as $g^{(1)}(z)$ vanishes except at the immediate neighbourhood of $R_i$, and will be ignored.

### 2.2. *Anisotropic barrier*

Now $u_0(r)$ and $v_l^{(n)}(r)$ in equation (5) are to be replaced by the corresponding *angle-dependent* potentials $u(\boldsymbol{r}')$ and $v_l^{(n)}(\boldsymbol{r}')$ and also will be referred to the body-fixed system of coordinates $r'(=r,\theta',\varphi')$, with origin at the same centre of mass. As already mentioned, if we confine our attention to spheroidal shapes of the nuclear surface, then

$$u(\boldsymbol{r}') = u_0(r)+u_2(r)\,\beta_{2,0}\,Y_{2,0}(\theta') \tag{6}$$

$$u_2(r) = \tfrac{3}{5}u_0(r)\left(\frac{R_c}{r}\right)^2 \tag{6.1}$$

and $R_c(=r_0\times(A-4)^{1/3}\times 10^{-13}\text{ cm})$ is the radius of a hypothetical nuclear sphere.

Now an axially symmetric $\alpha$-nuclear potential $v_l^{(n)}(\boldsymbol{r}')$ may be approximated in a way similar to that suggested by Gottfried (1956) for a static potential as

$$\begin{aligned} v_l^{(n)}(\boldsymbol{r}') &= v_l^{(n)}(r)-\bar{\epsilon}\{rp_l^{(n)}\,V'(r)+r\,V(r)\,p_l^{(n)\prime}(r)\} \\ &\quad\times Y_{2,0}(\theta')+(5/8\pi)\,\epsilon_l^2\{rp_l^{(n)}\,V'(r)+rV(r)\,p_l^{(n)\prime}(r)\} \end{aligned} \tag{7}$$

where $\bar{\epsilon} = \{\epsilon_l+(5/4\pi)^{1/2}\,\epsilon_l^2\}$.

Collecting the terms for the potentials the three-dimensional equation corresponding to equation (1) is

$$\left(-\frac{\hbar^2}{2\mu}\,\nabla^2_{r'}+H_{\text{int}}\right)\psi(\boldsymbol{r}') = E_l\,\psi(\boldsymbol{r}') \tag{8}$$

where

$$H_{\text{int}} = u(\boldsymbol{r}')+v_l^{(0)}(\boldsymbol{r}')+\frac{b^2\mu}{2\hbar^2}\,v_0^{(2)}(\boldsymbol{r}')\,\{u(\boldsymbol{r}')+v_l^{(0)}(\boldsymbol{r}')-E_l\}. \tag{8.1}$$

### 2.3. *Solutions*

In this section we find the solution of equation (8) by following, as mentioned in the Introduction, a method due to Christy and *adapted* to the present problem. The point of difference is that, in the case of a purely electrostatic barrier—for which the method has been discussed in detail by Fröman, there is no actual inner turning point because the function $K^2 (= u(\mathbf{r}') - E)$ in his paper has no discontinuity near the nuclear surface. On the other hand the existence of such a discontinuity for the function $F_l$ (corresponding to $K$ in equation (9) below—cf. equation (19)) necessitates some restriction to the lower limits of the integrals in the extremal problem:

$$\int_{\mathrm{P}}^{\mathrm{P}'} K\, \mathrm{d}s = \int_{R_c}^{r_{\mathrm{P}}} K\, \mathrm{d}r - \int_{R_c}^{r_{\mathrm{P}'}} K\, \mathrm{d}r + \ldots = \text{minimum} \tag{9}$$

where P′ is a point on the spheroidal nuclear surface $R(\theta')$, P is defined to be somewhere within the barrier region and $R_c$ is some average of $R(\theta')$. However, it is evident that, for $\theta' < 55°$, $R_c$ is less than $R_{\mathrm{P}'}$. For an electrostatic barrier this presents no difficulty because $K^2$ remains positive throughout except for $r > R_0$, the latter being the outer turning point $= 2(z-2)e^2/E$).

On the other hand, if $\alpha$-nuclear potential is superimposed on the Coulomb potential, there is an 'inner turning point' $R_i(\theta')$ (in three-dimensions), where the function $F_l^2$ (cf. equation (19) below) turns from a positive value outside to a negative value inside it. Clearly then the integration limit $R_c$ in equation (9) should nowhere be less than $R_i(\theta')$, otherwise $F_l^2$ will be negative which is not allowed. In contrast with $R(\theta')$ $(= R_c(1+\beta_{2,0} Y_{2,0}))$ which usually describes the nuclear surface, now $R_i(\theta')$ will depend on both $\beta_{2,0}$ and the non-locality parameter $\epsilon_l$, and, by analogy, we can write

$$R_i(\theta') = \bar{R}\{1 + \bar{\alpha}\, Y_{2,0}(\theta')\}. \tag{10}$$

Now it can be easily shown that

$$R_i(\theta') = R_a\{(1-\tfrac{1}{3}\sigma^2) + \tfrac{1}{3}\sigma^2 P_{2,0}(\cos\theta')\} \tag{10.1}$$

where $\bar{R}(= R_i(\theta' = \cos^{-1}(\tfrac{1}{3})^{1/2})$ is in general different from $R_c$ (cf. below equation (6.1) and occurring in equation (9)) and $\sigma$ is the eccentricity.

In the present context the lower limit in the extremal problem (9) should nowhere be less than $R_i(\theta')$ and hence should be chosen to be at least equal to (or greater than) the turning point $R_i(\theta')$ for all $\theta'$. This condition is obviously satisfied by the auxiliary sphere of radius $R_a$ (cf. equation (10.1)) which is also the semi-major axis of the generating ellipse for the surface $R_i(\theta')$.

The radial function for the $\alpha$-particle is denoted so far by only one quantum number, $l$. Now, for deformed nuclei, this will depend on other quantum numbers as well, specifying the intrinsic motions of the nucleons and the rotational and vibrational motions of the nucleus as a whole. The subscripts i and f will refer respectively to initial and final nuclei.

If we write out explicitly the products $v_0^{(2)}(\mathbf{r}') \times u(\mathbf{r}')$, $v_0^{(2)}(\mathbf{r}') \times v_l^{(0)}(\mathbf{r}')$ and $v_0^{(2)}(\mathbf{r}') \times E_l$ occurring in equation (8.1), and retain terms up to the order $\epsilon_l^2$ and $b^2$, then, after simplification, one finds the approximate solution of equation (8) as

$$B_{l,v,K_f}^{j_i}(r) = R_a \frac{U_l(E_l, r)}{U_l(E_l, R_a)} \left\{ \sum_{l'} h_{l'v} q_{l,l'}^{\,v}(a_l, \zeta_l) \right\} \tag{11}$$

with the boundary condition that the $\alpha$-wave function $\psi_0(\theta', \varphi')$ on the nuclear surface is

$$\psi_0(\theta', \varphi') = \sum_{l'} \sum_{v'} h_{l',v'}\, Y_{l',v'}(\theta', \varphi') \tag{11.1}$$

where the modified matrix

$$q_{l,l'}^{\nu}(a_l, \zeta_l) = \int_0^{\pi} \Theta_{l',\nu} \exp[-\{\zeta_l + a_l P_{2,0}(\theta)\}]\Theta_{l,\nu} \sin\theta \, d\theta \tag{12}$$

with $l$, $m$, $\nu$ being the alpha-angular momentum and its projections; $j$, $M$ and $K$ are the total angular momentum and its projections respectively, for the nucleus in the two systems of coordinates. Also

$$a_l = \{(5/4\pi)^{1/2} \mathscr{I}_1 + (\mathscr{I}_2 - \zeta_l)\} \tag{12.1}$$

$$\mathscr{I}_1 = \frac{(2\mu)^{1/2}}{\hbar} \int_{R_a}^{R_0} \Big[ [u_2(r)\, \beta_{2,0} - V(r)\, \bar{\epsilon}\{\bar{r} p_l^{(0)} - r p_l^{(0)\prime}(r)\} + \Delta K_1 - \Delta K_2]/2(u_0 - v_l^{(0)} - \Delta v_l - E_l)^{1/2} \Big] \, dr \tag{12.2}$$

where

$$\bar{r} = r \times 10^{13}/0{\cdot}574$$

$$\Delta v_l = \frac{b^2 \mu}{2\hbar^2} v_0^{(2)} \{u_0(r) - v_l^{(0)}(r) - E_l\} \tag{12.3}$$

$$\mathscr{I}_2 = \frac{(2\mu)^{1/2}}{\hbar} \int_{R_a}^{R_0} \frac{V(r)\, \bar{\epsilon}\{\bar{r} p_l^{(0)} - r p_l^{(0)\prime}(r)\}}{2\{u_0(r) - v_l^{(0)}(r) - \Delta v_l - E_l\}^{1/2}} \, dr \tag{12.4}$$

$$\zeta_l = \tfrac{1}{3}\sigma^2 \chi \left\{ \frac{kR_a}{\chi} \left(1 - \frac{3}{2} \frac{kR_a}{\chi}\right)(1 - K_a) \right\}^{1/2} + \mathscr{I}_2 \tag{12.5}$$

where

$$\chi = 2(Z-2)e^2(2\mu)^{1/2}/\hbar E_l^{1/2}, \qquad k = (2\mu E_l)^{1/2}/\hbar$$

$$K_a = \{v_l^{(0)}(R_a) + \Delta v_l(R_a)\}/\{u_0(R_a) - E_l\} \tag{12.6}$$

$$\Delta K_1 = (b^2\mu/2\hbar^2) p_0^{(2)} V(r) \times [u_2(r)\beta_{2,0} + \bar{r}\epsilon_l(u_0 - E_l) - \epsilon_l V(r) \times \{2\bar{r} p_l^{(0)} - r p_l^{(0)\prime}(r)\}] \tag{12.7}$$

$$\Delta K_2 = (b^2\mu/2\hbar^2) r\, V(r)\, \epsilon_l\, p_0^{(2)\prime}(r)(u_0 - v_l^{(0)} - E_l). \tag{12.8}$$

Now $U_l$ occurring in equation (10) is obtained by solving equation (5) by the well-known WKB method as

$$U_l = U_0 \exp\left\{\frac{l(l+1) y_0 I_0}{\chi}\right\} \tag{13}$$

where

$$y_0 = (Z-2)e^2/E_l^{1/2}$$

$$I_0 = \int_{R_a}^{R_0} \frac{\{1 + (\mu b^2/2\hbar^2)(g^{(0)} - p_0^{(2)})\, V(r)\}}{r^2\{u_0 - V(r) g^{(0)} - \Delta v_0 - E_l\}^{1/2}} \, dr. \tag{13.1}$$

We can now transform the solution $B_{l,\nu,K_f}^{j_i}(r)$ into the corresponding $\alpha$-wave function $S_{l,m,K_f}^{j_i}(r)$ in the laboratory system of coordinates by using the rotation matrix as

$$S_{l,m,K_f}^{j_i}(r) = R_a \left\{\frac{U_l(E_l, r)}{U_l(E_l, R_a)}\right\} \sum_{\nu} (-)^{j_f - j_i + \nu} (j_i, l; K_f + \nu, -\nu | j_f, K_f) \times \sum_{l'} h_{l'\nu} q_{l',l}^{\nu}(a_l, \zeta_l) \tag{14}$$

where for the Clebsch–Gordan addition coefficient we have used the notation following Condon and Shortley (1935).

## 3. Angular distributions

The probability per second per unit solid angle in the space-fixed system of coordinates that an $\alpha$-particle leaks out in the direction $(\theta, \varphi)$ leaving the daughter nucleus in the state $(j_f, K_f)$ is obtained by summing over $M_f$ the following expressions:

$$W(\theta) = (1/2\pi)(2E_l/\mu)^{1/2} \sum_{M_f} \Big| \sum_{l} (j_f, l; M_f, M_i - M_f | j_i, M_i) S_{l,m,j_f}^{j_i}(r)\, \Theta_{l,m}(\theta) \Big|^2_{r\to\infty} \quad (15)$$

where $\Theta_{l,m}(\theta)$ is the $\theta$-dependent part of the spherical harmonic $Y_{l,m}(\theta, \varphi)$. On using equations (14) and (13) in equation (15) and then simplifying the products of three Clebsch–Gordan coefficients in terms of the Racah associated coefficients and expanding the product of two spherical harmonics in terms of spherical harmonics, equation (15) is simplified. The resulting expression is then averaged with respect to the initial $M_i$ values. Now $M_i$ is involved in the Clebsch–Gordan coefficient only, and also recall that the vector addition coefficient is always equal to 1 for $L = 0$, and we take the average for $L = 2$, with respect to the initial distribution of $M_i$ values, of the expression

$$(L, 2; M_i, 0 | j_i, M_i) = \frac{6 j_i^2 f}{\{(2j_i - 1)2j_i(2j_i + 2)(2j_i + 3)\}^{1/2}} \quad (16)$$

where

$$f = \frac{1}{j_i} \langle M_i^2 - \tfrac{1}{3} j_i(j_i + 1) \rangle \quad (16.1)$$

which measures the nuclear polarization and vanishes for non-polarized parent nuclei. Now considering here only the ground state transitions, and on using equation (16), we can write equation (15) finally as

$$W(\theta) = W_0\{1 + \bar{f} A_2(j_i, j_f)\, P_2(\cos\theta) + \dots\} \quad (17)$$

where $\bar{f}$ is the average of $f$, $W_0$ is the isotropic angular distribution of the $\alpha$-particles and the desired formula for the anisotropy coefficient $A_2(j_i, j_f)$ is obtained as:

$$A_2(j_i, j_f) = (-)^{j_f - j_i} F(j_i) \Big\{ \sum_{l} \sum_{l'} \gamma_{l,l'}(l, l'; 0, 0 | 2, 0)\, W(l, j_i, l', j_i; j_f, 2) \omega_l \omega_{l'} \Big\} \Big/ \sum \omega_l^2 \quad (18)$$

where

$$F(j_i) = \frac{6(5)^{1/2} j_i^2 (2j_i + 1)^{1/2}}{\{(2j_i - 1)2j_i(2j_i + 2)(2j_i + 3)\}^{1/2}} \quad (18.1)$$

$$\gamma_{l,l'} = (-)^{(2 - l + l')}\{(2l + 1)(2l' + 1)\}^{1/2} \cos\left\{\frac{l(l+1) - l'(l'+1)}{\chi}\right\} \quad (18.2)$$

$$\omega_l = \exp\left\{-\frac{l(l+1) y_0 I_0}{\chi}\right\}(j_i, l; K_f, 0 | j_f, K_f) \Big\{ \sum_{l'} h_{l'\nu}\, q_{l,l'}^{\nu}(a_l, \zeta_l) \Big\} \quad (18.3)$$

and $q_{l,l'}^{\nu}$ is given by equation (12) and $W(l, j_i, l', j_i; j_f, 2)$ is the Racah coefficient (cf. Biedenharn *et al.* 1952).

## 4. Results

We now apply equation (18) to the $^{237}$Np nuclei. To evaluate the anisotropy coefficient $A_2$ we take advantage of the use of a computer; the integrals involved are worked out by using Simpson's rule in a slightly different form:

$$\int \sim \; = \tfrac{1}{3}d\{(f_0+2{\cdot}5f_{2n})+4(f_1+\ldots+f_{2n-1})+2(f_2+\ldots+f_{2n-2})\}$$

where $d$ is the strip width. We have taken the number of strips as 100 for the entire range of the integral, whereas, for calculating $g^{(0)}(z)$, $g^{(2)}(z)$ (cf. equation (4.2)), one obtains sufficiently accurate values with only 60 strips.

Now we calculate the values of $R_a$, $\bar{R}$ and $\sigma$ involved in the formula for $A_2$. Referring to equation (10), since the effective deformation parameter $\bar{\alpha}$ depends on both $\beta_{2,0}$ and $\epsilon_l$ it would not be surprising if, for some values of $\epsilon_l$ and for a given $\beta_{2,0}$, the barrier becomes spherical although $\beta_{2,0} \neq 0$. The value of $R_i(\theta)$ is obtained by solving by the iterative method the equation

$$F_l(r,\theta') = \frac{(2\mu)^{1/2}}{\hbar}(H_{\text{int}}-E_l)^{1/2} = 0 \tag{19}$$

where $H_{\text{int}}$ is given in equation (8). Now we give in table 1 the calculated values of the semi-major and semi-minor axes (i.e. $R_a$ and $R_{\min}$) and $\bar{R}$ of the nuclear spheroid for different values of $\epsilon_0$.

**Table 1.**

| Parent nucleus | Empirical $\beta_{2,0}$ | Non-local parameter $\epsilon_0$ | Inner turning point $R_i(\theta)$ (fm) $R_i(0 \text{ or } \pi)$ | $R_i(\pi/2)$ | Eccentricity $\sigma$ | Effective deformation $\bar{\alpha}$ |
|---|---|---|---|---|---|---|
| | | 0·150 | 9·032 | 8·987 | 0·099 | 0·005 |
| | | 0·125 | 9·019 | 8·995 | 0·074 | 0·003 |
| | | 0·050 | 8·981 | 9·015 | 0·087 | −0·002 |
| | 0·24 | 0·000 | 8·957 | 9·028 | 0·125 | −0·004 |
| | | −0·050 | 8·939 | 9·039 | 0·149 | −0·006 |
| $^{237}$Np | | −0·125 | 8·904 | 9·054 | 0·181 | −0·009 |
| | | −0·150 | 8·894 | 9·060 | 0·191 | −0·010 |
| | | 0·125 | 9·114 | 8·952 | 0·188 | 0·019 |
| | | 0·050 | 9·077 | 8·967 | 0·155 | 0·013 |
| | −0·24 | 0·000 | 9·053 | 8·980 | 0·127 | 0·009 |
| | | −0·050 | 9·029 | 8·991 | 0·091 | 0·005 |
| | | −0·125 | 8·995 | 9·007 | 0·050 | −0·001 |

Calculations are performed with range of non-locality $b = 0{\cdot}7$ fm and $\alpha$-energy equal to 4·787 MeV for the ground-state transitions in $^{237}$Np nuclei. The mean radius $\bar{R}(=R_i\{\cos^{-1}(\frac{1}{3})^{1/2}\})$ is found to be the same for all the cases and is equal to $9{\cdot}003\times10^{-13}$ cm.

As shown in table 1, the effective deformation $\bar{\alpha}$, is quite small compared with $\beta_{2,0}$. It is therefore evident that $R_c$ occurring in equation (6.1) (and also in the relation preceding equation (10)) is smaller than the mean radius $\bar{R}(=R_i(\cos^{-1}(\frac{1}{3})^{1/2})$ in equation (10)) for a given semi-major axis of the nuclear spheroid.

Next we calculate the anisotropy coefficient. As already stated, the sign of $Q$ for $^{237}$Np being unknown, calculations are given for both positive and negative signs of $Q$. The value of $\beta_{2,0}$ is chosen from a reference to the values given by Bell and co-workers (1960) for the region with mass number $A > 225$. In the relation $R_c = r_0(A-4)^{1/3}\times10^{-13}$ cm, it is possible, as already mentioned, to take $r_0 = 1{\cdot}20$.

Somewhat higher values of $r_0$ are also admissible. Since the non-local deformation parameter $\epsilon_0$ is unknown, a plausible range of values is chosen.

**Table 2.**

| Parent nucleus | $\beta_{2,0}$ | Non-local deformation $\epsilon_0$ | $\zeta_0$ Equation (12.5) | $a_0$ Equation (12.1) | $q_{0,0}$ | $q_{2,0}$ Equation (12) | $A_2(\frac{5}{2}, \frac{5}{2})$ Equation (18) |
|---|---|---|---|---|---|---|---|
| | | 0·150 | 0·103 | −1·440 | 1·255 | 0·900 | 1·53 |
| | | 0·125 | 0·072 | −1·087 | 1·136 | 0·611 | 1·17 |
| | | 0·050 | 0·023 | 0·061 | 1·000 | −0·027 | −0·06 |
| | 0·24 | 0·000 | 0·035 | 0·497 | 1·024 | −0·210 | −0·45 |
| | | −0·050 | 0·079 | 0·760 | 1·055 | −0·316 | −0·66 |
| | | −0·125 | 0·166 | 0·799 | 1·060 | −0·331 | −0·69 |
| | | −0·150 | 0·183 | 0·873 | 1·072 | −0·361 | −0·74 |
| $^{237}Np$ | | | | | | | |
| | | 0·125 | 0·095 | −2·210 | 1·710 | 1·833 | 2·13 |
| | | 0·050 | 0·028 | −1·228 | 1·178 | 0·719 | 1·33 |
| | −0·24 | 0·000 | 0·010 | −0·601 | 1·039 | 0·300 | 0·64 |
| | | −0·050 | 0·033 | −0·219 | 1·005 | 0·101 | 0·24 |
| | | −0·125 | 0·027 | −0·037 | 1·000 | 0·017 | 0·04 |
| | | −0·250 | 0·138 | 0·137 | 1·000 | −0·060 | −0·14 |
| | −0·20 | −0·250 | 0·151 | 0·202 | 1·000 | −0·088 | −0·20 |

By actual calculation it is found that $\zeta_l$ and $a_l$ involved in $q_{l,0}$ are not appreciably different from $\zeta_0$ and $a_0$ respectively, unless $l > 4$. In table 2, for calculating both $q_{2,0}$ and $q_{0,0}$ it is therefore sufficient to take $l = 0$ for both $\zeta_l$ and $a_l$. Also, for the even-parity transitions, $\nu = 0$, and the notation is simplified by dropping it.

## 5. Discussion

Our expression for the matrix element (cf. equation (12)) is quite different from that obtained earlier on the assumption of a purely electrostatic barrier. The matrix $q_{l,l'}^{\nu}$ is now a function of the $l$-dependent quantities $\zeta_l$ and $a_l$. It is also remarkable that the sign of the exponent in equation (12) now becomes *negative*. It may be seen from table 2 that, for positive values of $\beta_{2,0}$, the non-local parameter $\epsilon_0 > 0{\cdot}054$ if $A_2$ is to be *positive*. On the other hand, *for negative Q also*, and over a wide range of values of $\epsilon_0$, the anisotropy coefficient remains *positive*, *in contrast with the earlier expectation on the hypothesis of the electrostatic barrier*. This is, however, not surprising. Qualitatively it may be seen that the thinning out or lowering of the electrostatic barrier at the tips of the oblate nucleus would be more than compensated by a corresponding decrease in the *attractive* $\alpha$-nuclear interaction part of the barrier, because, the latter having a negative exponent in the form factor, has a much steeper gradient than the $1/r$ variation of the *isotropic* electrostatic potential.

It is, however, necessary to mention in this connection that the said compensation of the electrostatic barrier by the attractive $\alpha$-nuclear interaction potential may not be always complete, since what we have stated above relates only to the spherically symmetric components of the two potentials comprising the barrier. In addition, each part has terms due to spherical asymmetry of the nuclear surface, determined by the charge deformation parameter $\beta_{2,0}$ on one hand and by the non-locality parameter $\epsilon_l$ on the other. The relative signs of $\beta_{2,0}$ and $\epsilon_0$ are also very important

Hence the resultant shape of the barrier will be determined by a number of factors and the sign of the anisotropy coefficient will be known only by actual calculations. As can be seen from the trend of values of $A_2$ given in table 2, for oblate nuclei, the sign of $A_2$ may as well be negative only if $\epsilon_0$ would obtain quite large negative values, i.e. considerably less than $-0{\cdot}125$.

Before concluding, a few words about the coefficient in the expansion of $\psi_0(\theta, \varphi)$ (cf. equation (11.1)) are necessary. Some authors, for example Fröman, Strutinskii, and Rostovskii, assumed a constant $\alpha$-wave function on the nuclear surface, i.e., $h_{l',0} = 0$ for $l' \neq 0$. On the other hand, an angle-dependent $\alpha$-wave function on the nuclear surface has been taken by Mang and others. So far as the $\alpha$-angular distributions are concerned, the values of the coefficients $h_{2,\nu}$ are expected to be relatively small compared with the basic terms $h_{0,0}$ in $\psi_0(\theta, \varphi)$. If it were not so, the effects of angle dependence would either reduce the anisotropy to an insignificant value or would predominate over the effects of the barrier, thus giving a negative value for the anisotropy coefficient in the case of prolate nuclei. Either possibility is contrary to the experimental findings. Hence, the inclusion in our calculations of some plausible values of the quadrupole terms of $\psi_0(\theta, \varphi)$ should not affect our results appreciably.

## 6. Conclusions

From the present discussion one is led to conclude as follows. *Not only for the prolate but also for the oblate nuclei, $\alpha$-emissions should occur preferentially along the nuclear symmetry axis.* This conclusion is of course based on the presumption that, if $Q$ is positive, $\epsilon_0$ should be greater than $0{\cdot}054$ and, if $Q$ is negative, then $\epsilon_0$ must not be less than $-0{\cdot}150$. These limits in the values of $\epsilon_0$ are quite reasonable. As can be seen from table 2, with a positive $\beta_{2,0}$, if $\epsilon_0$ is less than $0{\cdot}05$, then the anisotropy coefficient becomes *negative*, which is contradicted by the experimental results. On the other hand for oblate nuclei, values of $\epsilon_0$ that would give a *negative* value of $A_2$, such that it is of the same order as observed for $^{237}$Np nuclei (i.e. $W(\theta)/W_0 = 1{\cdot}1$ at least), would require that $\epsilon_0$ is less than $-0{\cdot}25$. Such a large value of $\epsilon_l$ for $l = 0$ seems to be very improbable.

The main contribution of the present work is that the experiments of Hanauer and co-workers can now be understood without difficulty regardless of the sign of $Q$. Furthermore if $Q$ is positive we obtain that the non-local parameter $\epsilon_0$ should have a value in the range from $0{\cdot}054$ to $0{\cdot}060$ to be consistent with the measured anisotropy coefficient.

In applying the present considerations about the barrier to the problem of hindrance factors of $\alpha$-transitions in deformed nuclei in the range $92 \leqslant Z \leqslant 100$ which are known to be prolate, this indication about the value of $\epsilon_0$ will be useful.

Finally, it may be pointed out that if the anisotropy coefficient is measured for some $\alpha$-active nuclei for which $Q$ is indicated to be *negative* (e.g. $^{227}$Ac nuclei) the result should be quite significant. As our present considerations show that, in all probability, $A_2$ should be positive for oblate nuclei also, in contrast with the expectation from the hypothesis of a purely electrostatic barrier, it is expected that the experiments on $^{227}$Ac nuclei will provide conclusive evidence regarding the nature of the potential barrier in $\alpha$-decay theory.

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